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2009 J. Phys. A: Math. Theor. 42 085203

(http://iopscience.iop.org/1751-8121/42/8/085203)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **42** (2009) 085203 (11pp) doi:10.1088/1751-8113/42/8/085203

The Cauchy problem for the second member of a P_{IV} hierarchy

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Received 16 October 2008, in final form 21 December 2008
Published 30 January 2009
Online at stacks.iop.org/JPhysA/42/085203

Abstract

A rigorous method, the inverse monodromy transform, for studying the Riemann–Hilbert (RH) problem associated with the classical Painlevé equations, P_I–P_{VI}, is applied to the second member of a fourth Painlevé hierarchy. We show that the Cauchy problem for the second member of this P_{IV} hierarchy admits, in general, a global meromorphic solution in x. Moreover, for a particular choice of the monodromy data the associated RH problem can be reduced to a set of scalar RH problems and a special solution which can be written in terms of the Airy function is obtained.

PACS numbers: 02.30.Hq, 02.30.Gp, 02.30.Zz
Mathematics Subject Classification: 34M40, 34M50, 34M55, 34M05

1. Introduction

In this paper, we will apply the inverse monodromy transform (IMT) method to the second member of a P_{IV} hierarchy. This method is an extension of the inverse scattering transform (IST) for partial differential equations (PDE) to ordinary differential equations (ODE). The IMT can be thought of as a nonlinear analogue of Laplace's method used for finding the solution of linear ODE's. Flashka and Newell [1], and in a series of articles Jimbo, Miwa and Ueno [2], considered Painlevé equations as isomonodromic conditions for linear systems of ordinary differential equations having both regular and irregular singular points. Solving such an initial value problem is basically equivalent to solving an inverse problem for an associated isomonodromic linear equation. The inverse problem can be formulated in terms of the monodromy data which can be obtained from the initial data. Flashka and Newell [1] applied this method to P_{II} and to a special case of P_{III}, and they formulated the inverse problem in terms of a system of singular integral equations. In [2], the inverse problem is solved in terms of formal infinite series uniquely determined in terms of certain monodromy data.

Ablowitz and Fokas [3], and Fokas, Mugan and Ablowitz [4] formulated the inverse problems for P_{II}, and P_{IV}, P_V respectively, in terms of a matrix, singular, discontinuous Riemann–Hilbert (RH) boundary value problem defined on a complicated self-intersecting contour. A rigorous methodology for studying the RH problems appearing in the IMT was introduced by Fokas and Zhou [5], and they showed that the Cauchy problems for P_{II} and P_{IV} in general admit global solutions meromorphic in x. The above rigorous methodology was applied to P_I, P_{III}, P_V in [6], and to P_{VI} in [7]. In the recent monograph by Fokas, Its, Kapaev and Novokshenov [8] the inverse monodromy transform for P_I–P_V is discussed in great detail.

Equations P_I–P_{VI} are of course second order ODE's. The original classification programme of Painlevé foresaw a step-by-step classification of equations having the Painlevé property: after second-order, then third-order, then fourth-order and so on. Much current interest in the Painlevé equations derives from the important observation in [9] of a link between completely integrable PDE's and ODE's having the Painlevé property. Given that sitting above completely integrable PDE's such as the Korteweg–de Vries (KdV) and modified Korteweg–de Vries (mKdV) equations are their respective hierarchies, the way was then open to the derivation of hierarchies of higher order analogues of the Painlevé equations. However, even though Airault derived a P_{II} hierarchy (i.e., having P_{II} as the first member) almost 30 years ago [10] (see also [1]), it is only within the last 10 years or so that interest in Painlevé hierarchies has really taken off. Topics studied have included lifting up to higher order members of the hierarchy properties of the Painlevé equations themselves, for example, Bäcklund and auto-Bäcklund transformations, Hamiltonian structures, coalescence limits and special integrals. In the present paper we prove the existence of a globally meromorphic solution for a member of a Painlevé hierarchy which is not the standard one (as obtained from the 3-reduced KP hierarchy) related to (1+1)-dimensional evolution equations that correspond to non-isospectral scattering problems. This then provides evidence that the equations contained in such non-standard hierarchies are indeed of Painlevé type. The equation considered is the second member of a P_{IV} hierarchy obtained, using the approach developed in [11], and in[12].

The IMT method consists of two basic steps, the direct and inverse problems. The direct problem consists of establishing the analytic structure of the eigenfunction $\Phi(\lambda, x)$ of an associated linear equation in the complex λ-plane. In the case of the second member of the P_{IV} hierarchy, the linear ODE has a regular singular point at $\lambda = 0$, and an irregular singular point with rank $r = 3$ at $\lambda = \infty$. The eigenfunction $\Phi(\lambda, x)$, for large λ, has a unique asymptotic expansion in certain sectors of the λ-plane. According to the Stokes phenomenon these sectionally analytic eigenfunctions are related via Stokes matrices. In the neighbourhood of the regular singular point $\lambda = 0$, the solution can be obtained via convergent power series. The eigenfunction is normalized in the neighbourhood of $\lambda = 0$, and is related to the eigenfunction in the neighbourhood of $\lambda = \infty$ through the connection matrix. The set which consists of the entries of the Stokes matrices and connection matrix is called the set of monodromy data. Clearly, the monodromy data are independent of λ and also it can be shown that they are independent of x. The crucial part of the direct problem is to show that only four of the monodromy data are arbitrary. This can be shown by using the product condition around all singular points (consistency condition) and certain equivalence relations. Hence, for given four initial data for the second member of the P_{IV} hierarchy the four independent monodromy data can be obtained. In the inverse problem, a matrix RH problem over a self-intersecting contour can be formulated by using the results obtained from the direct problem. The jump matrices for the RH problem are uniquely defined in terms of the monodromy data. The RH problem is discontinuous at the points of the discontinuities of the associated linear problem. These discontinuities can be avoided by inserting the circle around $\lambda = 0$ and performing a small clockwise rotation. The new RH problem is continuous and equivalent to a certain

Fredholm integral equation. Once the solution of the new RH problem is obtained, the solution of the original one can easily be established. In order to have a regular RH problem, we choose the parameters of the second member of the P_{IV} hierarchy. However, this is without loss of generality since there exist Schlesinger transformations [13] which shift the parameters.

Since the eigenfunction $\Phi(\lambda, x)$ is defined as the solution of the RH problem, once the solution of the RH problem is obtained the associated linear ODE can be used for obtaining the solution u of the second member of the P_{IV} hierarchy. This procedure parameterizes the general solution of the second member of the P_{IV} hierarchy in terms of the relevant monodromy data and shows that the general solution is meromorphic in x. For certain choices of the monodromy data the RH problem can be solved in a closed form. We will show that for a particular choice of the monodromy data, the solution of the second member of the P_{IV} hierarchy can written in terms of the Airy function. An exhaustive investigation of all such cases will be given elsewhere.

The second member of the P_{IV} hierarchy corresponds to the system [12]:

$$\begin{aligned} u_{xx} &= 3uu_x - u^3 - 6uv - 2g_2xu + 2c_1(u_x - 2v - u^2) + 4\alpha_2, \\ v_{xx} &= 2\left[\frac{\left(uv + \frac{1}{2}v_x + c_1v - \alpha_2 + \frac{1}{2}g_2\right)^2 - \frac{1}{4}\beta_2^2}{v + \frac{1}{2}u^2 - \frac{1}{2}u_x + g_2x + c_1u}\right] - 2(uv)_x \\ &\quad - 2v\left(v + \frac{1}{2}u^2 - \frac{1}{2}u_x + g_2x\right) - 2c_1(v_x + uv). \end{aligned} \tag{1}$$

A scalar equation in u can be obtained by eliminating v between these two equations; we will refer to this scalar equation also as the second member of the P_{IV} hierarchy.

The second member of the P_{IV} hierarchy can also be obtained as the compatibility condition of the following system of linear equations [14]:

$$\frac{\partial\Phi}{\partial\lambda} = (B_2\lambda^2 + B_1\lambda + B_0 + B_{-1}\lambda^{-1})\Phi, \tag{2a}$$

$$\frac{\partial\Phi}{\partial x} = (A_1\lambda + A_0)\Phi, \tag{2b}$$

where

$$\begin{aligned} &B_2 = -2\sigma_3, \qquad B_1 = 2\begin{pmatrix} -c_1 & w \\ -v/w & c_1 \end{pmatrix}, \\ &B_0 = \begin{pmatrix} -(v + g_2x) & w(u + 2c_1) \\ -(v_x + uv + 2c_1v)/w & (v + g_2x) \end{pmatrix}, \\ &B_{-1} = \begin{pmatrix} -H & wL \\ -\left(H^2 - \frac{1}{4}\beta_2^2\right)/wL & H \end{pmatrix}, \\ &A_1 = -\sigma_3, \qquad A_0 = \begin{pmatrix} 0 & w \\ -v/w & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \end{aligned} \tag{3}$$

and

$$\begin{aligned} &u = -\frac{w_x}{w}, \qquad L \doteqdot \tfrac{1}{2}[2v + u^2 - u_x + 2g_2x + 2c_1u], \\ &H \doteqdot \tfrac{1}{2}[v_x + 2uv + 2c_1v - 2\alpha_2 + g_2], \end{aligned} \tag{4}$$

and g_2, α_2, β_2 are constants. Without loss of generality, we set $g_2 = 1$ and for simplicity of notation, we let $\alpha_2 = \alpha$, and $\beta_2 = \beta$.

2. Direct problem

The direct problem consists basically of establishing the analytic structure of the solution matrix Φ of (2) with respect to λ, in the entire complex λ-plane. To achieve this goal, we use (2*a*) which implies the existence of a regular singular point at $\lambda = 0$, and an irregular singular point with rank $r = 3$ at $\lambda = \infty$.

2.1. Solution of (2a) about $\lambda = 0$*:*

Since $\lambda = 0$ is a regular singular point of (2*a*), two linearly independent solutions $\Phi^0(\lambda) = \big(\Phi^0_{(1)}(\lambda), \Phi^0_{(2)}(\lambda)\big)$ in the neighbourhood of $\lambda = 0$ can be obtained via a convergent power series

$$\Phi^0(\lambda) = G_0\big(I + \hat{\Phi}^0_1\lambda + \hat{\Phi}^0_2\lambda^2 + \cdots\big)\left(\frac{1}{\lambda}\right)^{D_0}, \qquad \beta \neq n, \quad n \in \mathbb{Z}, \quad 0 < |\lambda| < \infty, \tag{5}$$

where

$$G_0 = \begin{pmatrix} \kappa_1 wL & \kappa_2 wL \\ \kappa_1\big(H + \frac{\beta}{2}\big) & \kappa_2\big(H - \frac{\beta}{2}\big)\end{pmatrix}, \qquad D_0 = -\frac{\beta}{2}\sigma_3, \tag{6}$$

where κ_1, κ_2 are constants with respect to λ, and $\hat{\Phi}^0_1$ satisfies $\hat{\Phi}^0_1 + \big[D_0, \hat{\Phi}^0_1\big] = G_0^{-1}B_0G_0$. If we impose the condition $\det G_0 = 1$, and use that $\Phi^0(\lambda)$ solves (2*b*), we find that κ_1 and κ_2 satisfy the following equations:

$$\kappa_1 = \frac{\rho}{wL}\exp\left[\int^x \frac{1}{L}\left(H + \frac{\beta}{2}\right)\mathrm{d}x'\right], \qquad \kappa_2 = -\frac{1}{\beta\rho}\exp\left[-\int^x \frac{1}{L}\left(H + \frac{\beta}{2}\right)\mathrm{d}x'\right], \tag{7}$$

where ρ is a constant with respect to x. If $\beta = n, n \in \mathbb{Z}$, then two linearly independent solutions are $\Phi^0_{(1)}(\lambda)$ and

$$\Phi^0_{(2)}(\lambda) = \tau(\ln\lambda)\Phi^0_{(1)}(\lambda) + \lambda^{-\beta/2}\chi(\lambda), \tag{8}$$

where $\chi = \chi_0 + \chi_1\lambda + \chi_2\lambda^2 + \cdots$. τ is a constant with respect to λ, and proportional to the coefficient of $\lambda^{2\beta-1}$ in $\Phi^0_{(1)}(\lambda)$. For example, when $\beta = \pm 1$

$$\tau\kappa_1 wL^2 = (\chi_0)_{11}\big[-L^2(v_x + uv + 2c_1v) + 2L(v + x)\big(H - \tfrac{1}{2}\big) - \big(H - \tfrac{1}{2}\big)^2(u + 2c_1)\big]. \tag{9}$$

Note that the logarithm will disappear if $\tau = 0$; when $\beta = \pm 1$ this implies

$$L^2(v_x + uv + 2c_1v) - 2L(v + x)\big(H - \tfrac{1}{2}\big) + \big(H - \tfrac{1}{2}\big)^2(u + 2c_1) = 0. \tag{10}$$

Equation (10) defines a three-parameter family of solutions of (1).

The monodromy matrix M_0 about $\lambda = 0$ is defined as

$$\Phi^0(\lambda\,\mathrm{e}^{2\mathrm{i}\pi}) = \Phi^0(\lambda)M_0, \qquad M_0 = \mathrm{e}^{-2\mathrm{i}\pi D_0}, \qquad \beta \neq n. \tag{11}$$

2.2. Solution of (2a) about $\lambda = \infty$*:*

$\lambda = \infty$ is an irregular singular point of (2*a*) with rank $r = 3$, and hence the solution of (2*a*) possesses a formal expansion of the form $\Phi(\lambda) \sim \Phi^\infty(\lambda) = \big(\Phi^\infty_{(1)}(\lambda), \Phi^\infty_{(2)}(\lambda)\big)$, as $\lambda \to \infty$, in certain sectors S^∞_j, $j = 1, \ldots, 6$ in the λ-plane. The formal expansion $\Phi^\infty(\lambda)$ near $\lambda = \infty$ is given by

$$\Phi^\infty(\lambda) = \hat{\Phi}^\infty(\lambda)\lambda^{D_\infty}\,\mathrm{e}^{Q(\lambda)} = \big(I + \hat{\Phi}^\infty_1\lambda^{-1} + \hat{\Phi}^\infty_2\lambda^{-2} + \cdots\big)\lambda^{D_\infty}\,\mathrm{e}^{Q(\lambda)}, \tag{12}$$

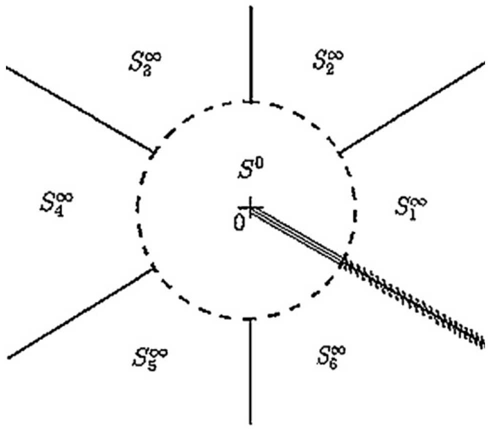

Figure 1. Sectors for the sectionally analytic function Φ.

where

$$\hat{\Phi}_1^\infty = \begin{pmatrix} \cdots & \frac{w}{2} \\ \frac{v}{2w} & \cdots \end{pmatrix}, \qquad D_\infty = \frac{1}{2}(2\alpha - 1)\sigma_3, \qquad Q(\lambda) = -\left(\frac{2}{3}\lambda^3 + c_1\lambda^2 + x\lambda\right)\sigma_3. \tag{13}$$

The relevant sectors S_j^∞, $j = 1, \ldots, 6$ are determined by $\mathrm{Re}\left[\left(\frac{2}{3}\lambda^3 + c_1\lambda^2 + x\lambda\right)\right] = 0$ and are given in figure 1. The non-singular matrices $\Phi_j(\lambda)$, $j = 1, \ldots, 6$ satisfy

$$\begin{aligned} \Phi_{j+1}(\lambda) &= \Phi_j(\lambda)G_j, && \lambda \in S_{j+1}^\infty, \quad j = 1, \ldots, 5, \\ \Phi_1(\lambda) &= \Phi_6(\lambda\,\mathrm{e}^{2\mathrm{i}\pi})G_6M_\infty^{-1}, && \lambda \in S_1^\infty, \end{aligned} \tag{14}$$

where the Stokes matrices G_j and the monodromy matrix M_∞ are given as

$$G_{2j-1} = \begin{pmatrix} 1 & a_{2j-1} \\ 0 & 1 \end{pmatrix}, \qquad G_{2j} = \begin{pmatrix} 1 & 0 \\ a_{2j} & 1 \end{pmatrix}, \qquad j = 1, 2, 3, \quad M_\infty = \mathrm{e}^{2\mathrm{i}\pi D_\infty} \tag{15}$$

and the sectors are

$$S_j^\infty : \frac{\pi}{6}(2j-3) \leqslant \arg z < \frac{\pi}{6}(2j-1), \qquad |z| > 0. \tag{16}$$

The entries a_j of the Stokes matrices G_j are constant with respect to λ.

Since Φ^0, Φ_1 are locally analytic solutions of the linear equation (2*a*), they are related with a constant (with respect to λ) matrix E which is called the connection matrix

$$\Phi_1(\lambda) = \Phi^0(\lambda)E, \qquad E = \begin{pmatrix} \alpha_0 & \beta_0 \\ \gamma_0 & \delta_0 \end{pmatrix}, \qquad \det E = 1, \tag{17}$$

where the $\det E = 1$ condition follows from the normalization of Φ^0 to have unit determinant. Branch cuts associated with the branch points $\lambda = 0, \infty$ are chosen along $0 \leqslant |\lambda| < 1$ and $1 < |\lambda| < \infty$, $\arg\lambda = -\pi/6$ respectively, and are indicated in figure 1. Clearly, the Stokes matrices G_j, $j = 1, \ldots, 6$, and the connection matrix E are constant matrices with respect to λ, but it can be shown that they are also independent of x [1].

Therefore, the analytic structure of the solution matrix Φ of (2) is characterized by the monodromy data $MD = \{a_1, a_2, a_3, a_4, a_5, a_6, \alpha_0, \beta_0, \gamma_0, \delta_0\}$. The monodromy data, MD satisfy the following product condition around all singular points, or consistency condition:

$$\prod_{j=1}^{6} G_j M_\infty^{-1} = E^{-1} M_0^{-1} E. \tag{18}$$

If Φ solves (2) with u satisfying the second member of the P_{IV} hierarchy, then $\bar{\Phi} = R^{-1}\Phi R$ where $R = \text{diag}(r^{1/2}, r^{-1/2})$ and r is a nonzero complex constant, also solves (2) with u satisfying the second member of the P_{IV} hierarchy. But the connection matrices $\bar{E}$ and the Stokes matrices $\bar{G}_j$ for $\bar{\Phi}$ are $\bar{E} = R^{-1}ER$, and $\bar{G}_j = R^{-1}G_j R$, respectively. Thus, r may be chosen to eliminate one of the parameters, e.g. $r = \beta_0$. Also, changing the arbitrary integration constant ρ (see equation (7)) amounts to multiplying $\Phi^0_{(1)}$ and $\Phi^0_{(2)}$ by arbitrary nonzero complex constants ϵ and ϵ^{-1}, respectively. This maps E to $\text{diag}(\epsilon, \epsilon^{-1})E$. Thus, ϵ may be chosen to eliminate one of the entries of the connection matrix E. The freedom in choosing E has no effect on the solution of the RH problem. Therefore, together with the consistency condition (18), and $\det E = 1$, these considerations imply that all the monodromy data can written in terms of four of them.

3. Inverse problem

In this section, we formulate a regular, continuous RH problem over the intersecting contours for the sectionally analytic function $\Psi(\lambda)$. $\Psi(\lambda)$ also depends on x; for simplicity in the notation we dropped x. We let $1/2 < \alpha < 3/2$, and $0 < \beta < 2$ in order to have integrable singularities at $\lambda = 0$ and $\lambda = \infty$. That is, in order to have a regular RH problem. However, this is without loss of generality, since there exist Schlesinger transformations [13] which shift the parameters α and β by half-integer and by integer, respectively. Hence, the Schlesinger transformations allow one to completely cover the parameter space.

Since $\hat{\Phi}^0$ and $\hat{\Phi}^\infty$ are holomorphic at $\lambda = 0, \infty$ respectively, in order to formulate a continuous RH problem, we insert the circle C_0 with radius $r < 1$ about the point $\lambda = 0$ (see figure 2). The jump matrices across the contours can be obtained from the definition of the Stokes matrices G_j (equations (15)) and the definition of the connection matrices E (equation (17)).

The jumps different from unity across the contours as indicated in figure 2 are given by

$$\begin{aligned}
&C_2 : \Phi_1 = \Phi_1 G_1, && AB : \Phi_1 = \Phi^0 E,\\
&C_3 : \Phi_3 = \Phi_2 G_2, && BC : \Phi_2 = \Phi^0 E G_1,\\
&C_4 : \Phi_4 = \Phi_3 G_3, && CD : \Phi_3 = \Phi^0 E G_1 G_2,\\
&DE : \Phi_4 = \Phi^0 E G_1 G_2 G_3, && C_5 : \Phi_5 = \Phi_4 G_4,\\
&EF : \Phi_5 = \Phi^0 E \prod_{j=1}^{4} G_j, && C_6 : \Phi_6 = \Phi_5 G_5,\\
&FA : \Phi_6 = \Phi^0 E \prod_{j}^{5} G_j, && C_1 : \Phi_1(z) = \Phi_6(z\,\mathrm{e}^{2\mathrm{i}\pi}) G_6 M_\infty^{-1}.
\end{aligned} \tag{19}$$

In order to define a continuous RH problem, we define a sectionally analytic function $\Psi(\lambda)$ as follows:

$$\Phi_j = \Psi_j\,\mathrm{e}^{Q(\lambda)}\lambda^{D_\infty}, \qquad j = 1, \ldots, 6, \qquad \Phi^0 = \Psi^0\,\mathrm{e}^{Q(\lambda)}\left(\frac{1}{\lambda}\right)^{D_0}, \tag{20}$$

where $Q(\lambda) = -\left(\frac{2}{3}\lambda^3 + c_1\lambda^2 + x\lambda\right)\sigma_3$, and $\Psi \to I$ as $\lambda \to \infty$.

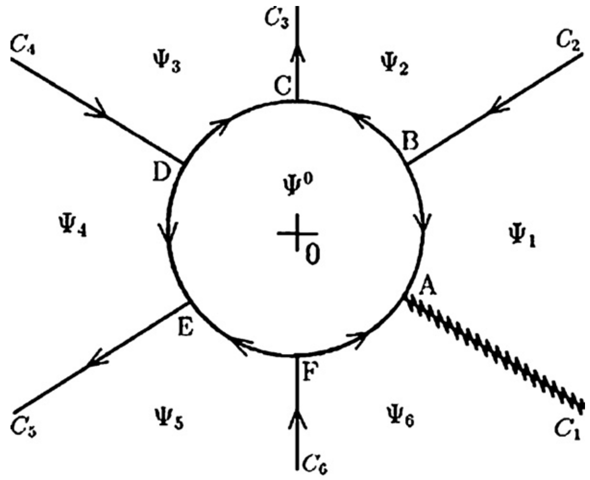

Figure 2. The contour for the RH problem.

The orientation as indicated in figure 2 allows the splitting of the complex λ-plane in + and − regions. Then (19) imply certain jumps for the sectionally analytic function Ψ which is represented by Ψ^0 and Ψ_j, $j = 1, \ldots, 6$, in the regions indicated in figure 2, and we obtain the following RH problem:

$$\Psi^+(\hat{\lambda}) = \Psi^-(\hat{\lambda})[\mathrm{e}^{Q(\hat{\lambda})} V \,\mathrm{e}^{-Q(\hat{\lambda})}] \quad \text{on} \quad C, \qquad \Psi = I + O\left(\frac{1}{\lambda}\right) \quad \text{as} \quad \lambda \to \infty, \tag{21}$$

where C is the sum of all the contours, and the jump matrices V are given in terms of the monodromy data as follows:

$$\begin{aligned}
&V_{C_2} = \lambda^{D_\infty} G_1{}^{-1} \lambda^{-D_\infty}, && V_{AB} = \lambda^{-D_0} E \lambda^{-D_\infty},\\
&V_{C_3} = \lambda^{D_\infty} G_2 \lambda^{-D_\infty}, && V_{BC} = \lambda^{D_\infty} (E G_1)^{-1} \lambda^{D_0},\\
&V_{C_4} = \lambda^{D_\infty} G_3^{-1} \lambda^{-D_\infty}, && V_{CD} = \lambda^{-D_0} E G_1 G_2 \lambda^{-D_\infty},\\
&V_{DE} = \lambda^{D_\infty} \left[E \prod_{j=1}^{3} G_j \right]^{-1} \lambda^{D_0}, && V_{C_5} = \lambda^{D_\infty} G_4 z^{-D_\infty},\\
&V_{EF} = \lambda^{-D_0} \left[E \prod_{j=1}^{4} G_j \right] \lambda^{-D_\infty}, && V_{C_6} = \lambda^{D_\infty} G_5^{-1} \lambda^{-D_\infty},\\
&V_{FA} = \lambda_+^{D_\infty} \left[E \prod_{j=1}^{5} G_j \right]^{-1} \lambda_+^{D_0}, && V_{C_1} = \lambda^{D_\infty} G_6 M_\infty^{-1} \lambda^{-D_\infty}.
\end{aligned} \tag{22}$$

Since we have the branch cut along the contour C_1, the subscript + appearing in the definition of V_{FA} indicates that we consider the relevant boundary values from + region, that is, $z_+ = |\lambda| \,\mathrm{e}^{2\mathrm{i}\pi}$.

By construction Ψ satisfies the continuous RH problem and this can be checked by the product of the jump matrices V at the intersection points of the contours. The product of the jump matrices at the intersection points B, C, D, E, F identically equals the identity matrix I, and at the point A equals I because of the consistency condition (18) of the monodromy data.

The RH problem (21) is equivalent to following Fredholm integral equation:

$$\Psi^{-}(\lambda) = I + \frac{1}{2i\pi}\int_C \frac{\Psi^{-}(\hat{\lambda})[V(\hat{\lambda})V^{-1}(\lambda) - I]}{\hat{\lambda} - \lambda}\,d\hat{\lambda}, \tag{23}$$

where C is the sum of all the contours. The Cauchy problem for the second member of the P_{IV} hierarchy always admits a global meromorphic solution in x. These solutions can be obtained by solving the associated RH problem of the form $\Psi^{+}(\hat{\lambda}) = \Psi^{-}(\hat{\lambda})[e^{Q(\hat{\lambda})}V\,e^{-Q(\hat{\lambda})}]$ where the jump matrices V are given in terms of the monodromy data, which are such that four of them are arbitrary. Once the solution Ψ of the associated RH problem is obtained, the solution u of the second member of the P_{IV} hierarchy is obtained from

$$u = -2\frac{\partial}{\partial x}\ln(\Psi_{-1})_{12}, \tag{24}$$

where

$$\Psi = I + \Psi_{-1}\lambda^{-1} + \Psi_{-2}\lambda^{-2} + \cdots, \qquad \text{as} \quad \lambda \to \infty, \tag{25}$$

and $(\Psi_{-1})_{12}$ is $(1, 2)$ entry of Ψ_{-1}.

For a special choice of the monodromy data, the jump matrix V of the RH problem (21) can be reduced to a triangular matrix, and hence the RH problem can be reduced to a set of scalar RH problems. The closed form solution of the set of scalar RH problems can be obtained by using the Plemelj formulae. We consider the following case; an exhaustive investigation of all such cases will be given elsewhere. Let

$$a_2 = a_3 = a_4 = 0, \qquad \text{and} \qquad \beta_0 = \gamma_0 = 0. \tag{26}$$

Without loss of generality, we let $E = I$. Then the consistency condition of the monodromy data (18) implies that

$$a_5 = -a_1 = a, \qquad a_6 = 0, \qquad 2\alpha - \beta - 1 = 2n, \qquad n \in \mathbb{Z}. \tag{27}$$

Let $n = 0$, and $\beta = 0$, then $\alpha = 1/2$, and the RH problem (21) is reduced to one along the contour C as indicated in figure 3, with an upper-triangular jump matrix

$$\Psi^{+}(\hat{\lambda}) = \Psi^{-}(\hat{\lambda})\begin{pmatrix} 1 & -a\,e^{-2q(\hat{\lambda})} \\ 0 & 1 \end{pmatrix}, \qquad \text{on} \quad C, \qquad \Psi = I + O\left(\frac{1}{\lambda}\right) \quad \text{as} \quad \lambda \to \infty, \tag{28}$$

where $q(\lambda) = \frac{2}{3}\lambda^3 + c_1\lambda^2 + x\lambda$.

Letting $\Psi = (\Psi_1, \Psi_2)$, the above RH problem reduces to the following set of scalar RH problems:

$$\Psi_1^{+} = \Psi_1^{-}, \tag{29a}$$

$$\Psi_2^{+} - \Psi_2^{-} = -a\,e^{-2q}\Psi_1^{-}. \tag{29b}$$

With the choice $\beta = 0$, the boundary condition on Ψ implies that

$$\Psi_1 = \Psi_1^{+} = \Psi_1^{-} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{30}$$

Then, using Plemelj formulae, the solution of (29b) is given as

$$\Psi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{1}{2i\pi}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\int_C \frac{a\,e^{-2q(\hat{\lambda})}}{\hat{\lambda} - \lambda}\,d\hat{\lambda}. \tag{31}$$

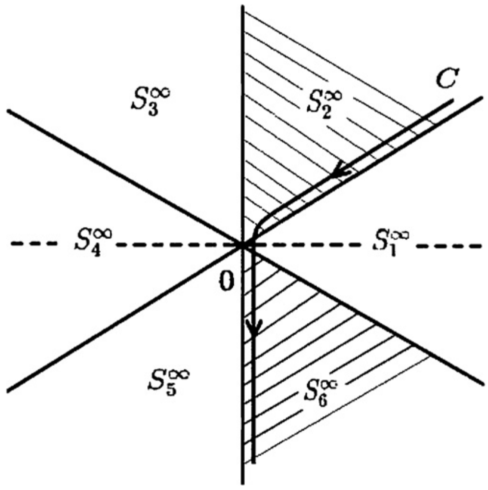

Figure 3. The contour for the integral representation of the Airy function.

Therefore, the solution of the RH problem (28) is

$$\Psi(\lambda) = \begin{pmatrix} 1 & \Lambda(\lambda) \\ 0 & 1 \end{pmatrix}, \qquad \Lambda(\lambda) \doteqdot -\frac{a}{2i\pi}\int_C \frac{e^{-2q(\hat\lambda)}}{\hat\lambda - \lambda}\,d\hat\lambda. \tag{32}$$

If one expands Λ in powers of $1/\lambda$, the coefficient of the $O(1/\lambda)$ term is the integral representation of the Airy function $Ai(-x)$ for $c_1 = 0$. Therefore, for $\beta = c_1 = 0$ and $\alpha = 1/2$, the solution u of the second member of the P_{IV} hierarchy is expressible rationally in terms of the Airy function (see equation (24)).

4. Derivation of the linear problem

In this section, we show that once the sectionally analytic function Ψ satisfying the RH problem (21) is known, then the coefficients A and B of the Lax pair (2) can be determined and hence the solution u of the second member of the P_{IV} hierarchy. Note that the sectionally analytic functions Φ and Ψ are defined as Φ^0, Φ_j and Ψ^0, Ψ_j, $j = 1, \ldots, 5$ respectively, and Φ and Ψ are related via (20).

Since Φ and Φ_λ admit the same jumps it follows that $B = \Phi_\lambda\Phi^{-1}$ is holomorphic in $\mathbb{C}/\{0\}$. Moreover, $\Phi \sim \exp\left[-\left(\frac{2}{3}\lambda^3 + c_1\lambda^2 + x\lambda\right)\sigma_3\right]\lambda^{\frac{1}{2}(2\alpha-1)\sigma_3}$, as $\lambda \to \infty$. Therefore, $B(\lambda) = B_2\lambda^2 + B_1\lambda + B_0 + B_{-1}\lambda^{-1}$. Equation (20), and $\Phi_\lambda = B\Phi$ give

$$\Psi_\lambda - (2\lambda^2 + 2c_1\lambda + x)\Psi\sigma_3 + \frac{1}{2\lambda}(2\alpha - 1)\Psi\sigma_3 = (B_2\lambda^2 + B_1\lambda + B_0 + B_{-1}\lambda^{-1})\Psi, \tag{33a}$$

$$\Psi_\lambda - (2\lambda^2 + 2c_1\lambda + x)\Psi\sigma_3 + \frac{\beta}{2\lambda}\Psi\sigma_3 = (B_2\lambda^2 + B_1\lambda + B_0 + B_{-1}\lambda^{-1})\Psi, \tag{33b}$$

near $\lambda = \infty$, and $\lambda = 0$ respectively. As $\lambda \to \infty$, Ψ has the expansion

$$\Psi = I + \Psi_{-1}\lambda^{-1} + \Psi_{-2}\lambda^{-2} + \Psi_{-3}\lambda^{-3} + \cdots. \tag{34}$$

Substituting (34) into (33a) yields

$$O(\lambda^2) : B_2 = -2\sigma_3, \tag{35a}$$

$$O(\lambda): B_1 = -2c_1\sigma_3 + 2[\sigma_3, \Psi_{-1}], \tag{35b}$$

$$O(1): B_0 = -x\sigma_3 + 2[\sigma_3, \Psi_{-2}] + 2[\sigma_3, \Psi_{-1}](c_1 I - \Psi_{-1}), \tag{35c}$$

$$O(\lambda^{-1}): B_1\Psi_{-2} + B_0\Psi_{-1} + B_{-1} = 2[\sigma_3, \Psi_{-3}] + \left[\tfrac{1}{2}(2\alpha - 1)I - 2c_1\Psi_{-2} - x\Psi_{-1}\right]\sigma_3. \tag{35d}$$

If we define w and v as

$$w = 2(\Psi_{-1})_{12}, \qquad v = 2w(\Psi_{-1})_{21}, \tag{36}$$

then (35b) implies

$$B_1 = 2\begin{pmatrix} -c_1 & w \\ -\frac{v}{w} & c_1 \end{pmatrix}. \tag{37}$$

Equations (35c) and (36) yield

$$(B_0)_{11} = -(B_0)_{22} = -(x + v). \tag{38}$$

Similar considerations imply that $A(\lambda) = A_1\lambda + A_0$. Equation (20), and $\Phi_x = A\Phi$ give

$$\frac{\partial \Psi}{\partial x} - \Psi\sigma_3\lambda = (A_1\lambda + A_0)\Psi. \tag{39}$$

Substituting (34) into (39) gives

$$O(\lambda): A_1 = -\sigma_3, \qquad O(1): A_0 = [\sigma_3, \Psi_{-1}], \tag{40a}$$

$$O(\lambda^{-1}): (\Psi_{-1})_x = [\sigma_3, \Psi_{-1}]\Psi_{-1} - [\sigma_3, \Psi_{-2}], \tag{40b}$$

$$O(\lambda^{-2}): (\Psi_{-2})_x = [\sigma_3, \Psi_{-1}]\Psi_{-2} - [\sigma_3, \Psi_{-3}]. \tag{40c}$$

Using $(\Psi_{-1})_{12}$ and $(\Psi_{-1})_{21}$ as given in equation (36), we find A_0 as given in equation (3). Using equation (40c) in (35c), we find

$$B_0 = -z\sigma_3 + 2c_1[\sigma_3, \Psi_{-1}] - 2(\Psi_{-1})_x, \tag{41}$$

and hence

$$(B_0)_{12} = w(u + 2c_1), \qquad (B_0)_{21} = -\frac{1}{w}(v_x + uv + 2c_1 v). \tag{42}$$

On the other hand, equations (42) and (35c) imply

$$2w(\Psi_{-1})_{22} - 4(\Psi_{-2})_{12} = w_x, \qquad 4(\Psi_{-2})_{21} - \frac{2v}{w}(\Psi_{-1})_{11} = \frac{1}{w}(v_x + uv). \tag{43}$$

Then, from equation (35d), we obtain

$$(B_{-1})_{11} = -(B_{-1})_{22} = -\tfrac{1}{2}(v_x + 2uv + 2c_1 v - 2\alpha + 1) \doteqdot -H. \tag{44}$$

As $\lambda \to 0$, equation (33b) implies

$$B_{-1} = \frac{\beta}{2}\Psi(0)\sigma_3[\Psi(0)]^{-1}, \tag{45}$$

thus

$$\det B_{-1} = -\frac{\beta^2}{4}, \qquad \text{and} \qquad \operatorname{tr} B_{-1} = 0. \tag{46}$$

These equations together with the expression for $(B_{-1})_{11}$ yields B_{-1} as given in equation (3).

Acknowledgments

UM is partially supported by The Scientific and Technological Research Council of Turkey (TÜBITAK) under grant number 108T977. The work of AP is supported in part by the Ministry of Education and Science of Spain under contract MTM2006-14603, the Spanish Agency for International Cooperation under contract A/010783/07, the Universidad Rey Juan Carlos and Madrid Regional Government under contract URJC-CM-2006-CET-0585, and the Junta de Castilla y León under contract SA034A08. The authors thank the anonymous referees for their helpful and illuminating suggestions.

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